BI-TP 97/19
June 1997

On QCD Thermodynamics with Improved Actions *

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We discuss recent advances in the calculation of thermodynamic observables using improved actions. In particular, we discuss the calculation of the equation of state of the $SU(3)$ gauge theory, the critical temperature in units of the string tension, the surface tension and the latent heat at the deconfinement transition. We also present first results from a calculation of the equation of state for four-flavour QCD using an $\mathcal{O}(a^2)$ improved staggered fermion action and discuss possible further improvements of the staggered fermion action.

1. Introduction

The idea that strongly interacting hadronic matter undergoes a phase transition to a new phase, the quark-gluon plasma, has been around for a long time. More than 10 years ago lattice calculations have given first direct evidence for the existence of such a phase transition and opened the way for its detailed quantitative analysis. Many of the non-perturbative features of finite temperature QCD, which originally emerged from perturbative studies have since then been analyzed in lattice calculations. However, many of these numerical studies also had to remain on a qualitative level for a long time because simulations on coarse lattices were hampered by large discretization errors and calculations at smaller lattice spacing suffered from poor statistical accuracy.

The discretization scheme proposed originally by K. Wilson for gauge theories [1] does for a finite lattice cut-off, a, introduce systematic $\mathcal{O}(a^2)$ deviations from the continuum formulation. It is well known that this introduces severe problems in thermodynamic calculations, which in the past made a direct determination of the physics in the continuum limit difficult. Already the thermodynamics of free Bose (gluon) or Fermi (quark) gases deviates strongly from the continuum ideal gas results when calculated on coarse Euclidean lattices [2]. This problem carries over to QCD, where bulk thermodynamic observables like energy density and pressure do rapidly come close to the non-interacting ideal gas limit above the deconfinement phase transition. The calculation of these quantities is thus expected to suffer from similarly strong cut-off effects as the ideal gas, despite the fact that the high temperature plasma phase does in many respects still show strong non-perturbative properties, characterized e.g. by thermal screening masses and quasi-particle excitations.

At high temperature the relevant contributions to thermodynamic observables result from momenta which are of the order of the temperature, *i.e.* $\langle p\rangle \sim 3T$. However, in lattice calculations T is fixed through the temporal extent, N_τ, of the lattice and the cut-off, a, *i.e.* $T \equiv 1/N_\tau a$. As the computational effort in the calculation of bulk thermodynamic observables, which are of dimension T^4, increases approximately like N_τ^{10}, it is evident that the temporal extent of the lattice has to remain rather small in most thermodynamic calculations. The relevant momenta for e.g. lattices with temporal extent $N_\tau = 4$ are therefore close to the cut-off where the lattice and continuum dispersion relations differ strongly from each other. Indeed this is the origin of the well-known discrepancy between the energy density of an ideal gas calculated on a finite lattice ($\epsilon(N_\tau)$) and in the continuum ($\epsilon_{\rm SB}$) [2]. The cut-off dependence in finite temperature calculations thus shows up as a *finite size effect*, which should not be confused with the finite size dependence resulting from the spatial extent of the lattice. The latter is an infra-red effect and controls the approach to the thermodynamic limit.

In the next section we briefly discuss some improved actions, which recently have been used in thermodynamic calculations. In Section 3 we present results from a perturbative calculation of cut-off effects in the ideal gas (infinite temperature) limit. Results from numerical simulations of the $SU(3)$ gauge theory and four-flavour QCD are discussed in Sections 4 and 5. In Section 6 we suggest a new staggered fermion action suitable for thermodynamic calculations. Finally we give our conclusions in Section 7.

*Talk given at the International Workshop "Lattice QCD on Parallel Computers", Tsukuba, Japan, 10-15 March 1997; to be published in Nucl. Phys. B Supplements

2. Improved actions

During the last few years much progress has been made in dealing with the systematic discretization errors in lattice regularized quantum field theories. Various improved discretization schemes for the Lagrangian of QCD have been constructed and explored, which do show much less cut-off dependence than the prescription originally given by Wilson [3–5].

When formulating a discretized version of QCD one has a great deal of freedom in choosing a lattice action. Different formulations may differ by subleading powers of the lattice cut-off, which vanish in the continuum limit. This has, for instance, been used by Symanzik to systematically improve scalar field theories [6] and has then been applied to lattice regularized $SU(N)$ gauge theories [7,8]. In addition to the elementary plaquette term appearing in the standard Wilson formulation of lattice QCD larger loops can be added to the action in such a way that the leading $\mathcal{O}(a^2 g^0)$ deviations from the continuum formulation are eliminated and corrections only start in $\mathcal{O}(a^4 g^0, a^2 g^2)$. A simple class of improved actions is, for instance, obtained by adding planar loops of size (k,l) to the standard Wilson action (one-plaquette action)

$$S^{(k,l)} = \sum_{x,\nu>\mu} \Big(c_0^{(k,l)}\; W^{1,1}_{\mu,\nu}(x) + c_1^{(k,l)} W^{k,l}_{\mu,\nu}(x) \Big) \ , \quad (1)$$

where $W^{k,l}_{\mu,\nu}$ denotes the average of planar Wilson loops of size (k,l) and (l,k) in the (μ,ν) plane of a 4-dimensional lattice of size $N_\sigma^3 \times N_\tau$[9]. The coefficients c_i are, in general, functions of the gauge coupling g^2, *i.e.* $c_i \equiv c_i(g^2)$. In the limit $g^2 \to 0$ the two coefficients in Eq. 1 should fulfill the relation $c_0^{(k,l)}(0) = 1 - k^2 l^2 c_1^{(k,l)}(0)$ in order to insure the correct continuum limit. A specific choice of $c_1^{(k,l)}(0)$ allows to eliminate the leading $\mathcal{O}(a^2)$ corrections on the

tree level: $c_1^{(k,l)}(0) = -2/\big[k^2 l^2 (k^2 + l^2 - 2)\big]$

In the following we will present results for the tree level improved actions $S^{(1,2)}$ and $S^{(2,2)}$ which have been used recently for thermodynamic calculations [9–11]. These actions can be further improved by adding either additional loops, which would allow a systematic perturbative improvement in higher orders of g^2, or they may be improved non-perturbatively by decorating the expansion coefficients with tadpole improvement factors [12]

$$c_1^{(k,l)}(0) \to c_{1,tad}^{(k,l)} = u_0^{-2(k+l-2)} c_1^{(k,l)}(0)$$

Here the factor u_0 is determined self-consistently, for instance from plaquette expectation values [12], $u_0^4 = \frac{1}{6N_\sigma^3 N_\tau} \ \langle \sum_{x,\nu>\mu} (1 - \frac{1}{N} \mathrm{Re\ Tr} \,\square_{\mu\nu}(x)) \ \rangle$. Studies at $T=0$ have shown that this approach indeed leads to a strong reduction of the cut-off dependence even on rather coarse lattices. This is, of course, most efficient for observables which are sensitive to short distance scales. We will discuss here in how far this improvement scheme helps to improve finite temperature observables.

Another way of generating an improved action within the class of actions defined by Eq. 1 has been suggested by Y. Iwasaki [13] and has been extensively used recently for thermodynamic studies [14]. Here renormalization group (RG) methods have been used to generate an action close to the renormalized trajectory, which then has been projected onto the two parameter space defined by Eq. 1. This action is defined by the choice

$$c_1^{(1,2)} \equiv c_1^{\mathrm{RG}} = -0.662 \ . \quad (2)$$

Finally we should mention the class of fixed point actions, which has been constructed recently [3] and which also have been used for thermodynamic calculations [15,16]. The fixed point action reproduces the continuum dispersion relation and therefore does, by construction, also eliminate all cut-off dependences. These do return, when the fixed point action, which depends on an infinite number of parameters, is projected on to a limited parameter space. A frequently used truncated version is expressed in terms of powers of two basic Wilson loops, the plaquette, $W^{(1,1)}_{\mu,\nu}$, and a non-planar six-link Wilson loop, which we denote here by $W_{\mu,\nu,\rho}$. The action then reads

$$S^{\mathrm{FP}} = \sum_{x,\nu>\mu} \sum_{n=1}^{k} d_{0,n} \ \big[W^{1,1}_{\mu,\nu}(x)\big]^n + \sum_{x,\nu>\mu>\rho} \sum_{n=1}^{k} d_{1,n} \ \big[W_{\mu,\nu,\rho}(x)\big]^n \ , \quad (3)$$

here $W_{\mu,\nu,\rho}(x)$ denotes the average over the four

distinct paths of length six on the boundary of a 3-d cube. In the high temperature, ideal gas limit these truncated fixed point actions are indeed very similar to the ones defined by Eq. 1. The ideal gas limit receives contributions only from the $n = 1$ terms in the above sums, *i.e.* it is controlled by the two couplings $d_{0,1}$ and $d_{1,1}$ only. The ansatz given by Eq. 3 may also be used to define a tree level improved action by choosing $k = 1$ and $d_{0,1} = 1/3$, $d_{1,1} = 1/3$. This is similar to Eq. 1 with the planar six-link loop replaced by a *twisted* six-link loop.

3. The High Temperature Ideal Gas Limit

The importance of improved actions for thermodynamic calculations becomes evident immediately from an analysis of the high (infinite) temperature limit for a gluon gas on lattices with temporal extent N_τ. In the standard Wilson formulation the $\mathcal{O}(a^2)$ cut-off dependence in physical observables is reflected in large $\mathcal{O}(N_\tau^{-2})$ deviations from the continuum Stefan-Boltzmann limit. On lattices with temporal extent N_τ one finds, for instance, for the deviation of the energy density, $\epsilon(N_\tau)$, from the continuum result, $\epsilon_{\rm SB}$, [9][b]

$$\frac{\epsilon(N_\tau)}{\epsilon_{\rm SB}} = 1 + \frac{10}{21}\Big(\frac{\pi}{N_\tau}\Big)^2 + \frac{2}{5}\Big(\frac{\pi}{N_\tau}\Big)^4 + \mathcal{O}\big(N_\tau^{-6}\big) \quad (4)$$

The large $\mathcal{O}(a^2)$ corrections are eliminated by using the improved actions defined in Eq. 1. One indeed finds a strong reduction of the cut-off dependence relative to the standard Wilson formulation. In fact, although one has no control over the sub-leading corrections when setting up the systematic $\mathcal{O}(a^2)$ improvement it turns out that in many cases not only the $\mathcal{O}(a^2)$ corrections get eliminated but also the $\mathcal{O}(a^4)$ corrections get reduced. For instance, one finds [9]

$$\frac{\epsilon(N_\tau)}{\epsilon_{\rm SB}} = 1 + c_I \cdot \Big(\frac{\pi}{N_\tau}\Big)^4 + \mathcal{O}\big(N_\tau^{-6}\big) \quad (5)$$

with $c_I = 0.044(2)$ for the action $S^{(1,2)}$ and $c_I = 0.178(2)$ for $S^{(2,2)}$. This leads to a drastic reduction of cut-off errors as can be seen from Fig. 1. Of course, in this infinite temperature limit tadpole improvement does not affect the thermodynamics. In how far it leads to an improvement at finite temperatures will be discussed in the next section.

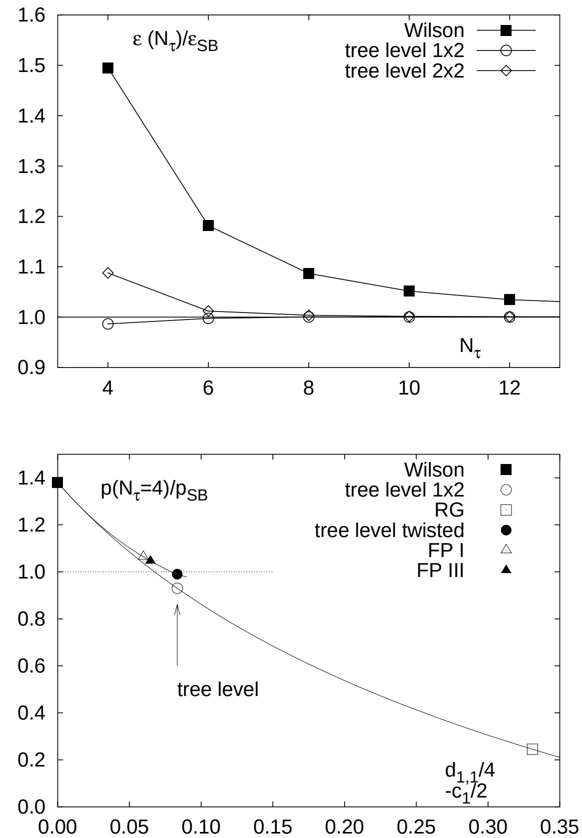

Figure 1. Cut-off dependence of the energy density and pressure in the infinite temperature, ideal gas limit. The upper figure shows results for some actions as function of the temporal lattice extent N_τ. In the lower figure results are shown for $N_\tau = 4$. For further discussions see the text.

In the general case of actions, which also include non-planar loops, it is more suitable to calculate the pressure, p, (or free energy density, $f = -p$) rather than the energy density. The former is proportional to the logarithm of the partition function, and thus is easily obtained in perturbative calculations. In the lower part of Fig. 1 we also show the ratio $p(N_\tau)/p_{\rm SB}$ for $N_\tau = 4$ for several actions defined through Eqs. 1 and 3 with varying coefficients c_1 and $d_{1,1}$, respectively. This also includes the case of the RG-improved action defined through

[b]The $\mathcal{O}(N_\tau^{-4})$ coefficient has been given incorrectly in [9]. It should read 2/5 rather than 2/6. We thank T. Scheideler for pointing this out to us.

Eq. 2 (open square). As can be seen this action leads to rather large cut-off dependent corrections in the high temperature limit. In this figure we also show results for two versions of the fixed point action[c] [15] (triangles) and the tree level improved actions including planar (open circle) and twisted (filled circle) six-link loops. As can be seen all these actions do lead to small deviations from the continuum result already on lattices with temporal extent $N_\tau = 4$.

A similar systematic improvement as discussed above for the gluonic sector can be achieved in the fermion sector [17]. We will present some results for four-flavour QCD in Section 5. In the next section we start with a discussion of results obtained for the thermodynamics of the pure SU(3) gauge theory using mainly tree level and tadpole improved actions [9–11].

4. SU(3) Thermodynamics

Only recently calculations with the standard Wilson action could be extended to lattices with sufficiently large temporal extent ($N_\tau = 6$ and 8) that would allow an extrapolation of lattice results for bulk thermodynamic quantities to the continuum limit [18,19]. Computationally the step from $N_\tau = 4$ to $N_\tau = 8$ is quite non trivial as the computer time needed to achieve numerical results with the same statistical significance on a two times larger lattice increases roughly like 2^{10}. It therefore is highly desirable to use improved actions, which suffer less from discretization errors, also for thermodynamic calculations.

We have analyzed the thermodynamics of the SU(3) gauge theory using the $\mathcal{O}(a^2)$ tree level ($u_0 \equiv 1$) and tadpole ($u_0 < 1$) improved actions defined in Eq. 1. As discussed above it seems to be plausible that improved actions will help to reduce the cut-off dependence of thermodynamic quantities in the high temperature plasma phase. It is, however, less evident that this also is of advantage for calculations close to the deconfinement phase transition where the physics is strongly dependent on contributions from infrared modes. In addition to a calculation of the equation of state an analysis of the cut-off dependence of the critical temperature itself as well as properties of the first order deconfinement transition like the surface tension and latent heat are therefore of interest.

4.1. The critical temperature

When comparing the cut-off dependence of thermodynamic observables one, of course, has to make sure that these are compared at the same value of the cut-off. On lattices with fixed temporal extent this amounts to a comparison at the same value of the temperature, which, for instance, can be defined in terms of the critical temperature for the deconfinement transition, *i.e.* T/T_c. Also the determination of this temperature scale is influenced by the finite lattice spacing and will contribute to the overall cut-off dependence of thermodynamic observables. An indication for the size of cut-off dependence in the definition of a temperature scale can be deduced from a calculation of the critical temperature in units of $\sqrt{\sigma}$. The ratio $T_c/\sqrt{\sigma}$ has been studied in quite some detail at different values of the cut-off for the $SU(3)$ gauge theory with the Wilson action. The $\mathcal{O}(a^2)$ dependence has clearly been seen in these calculations. An extrapolation of these results to the continuum limit yields[d]

$$\frac{T_c}{\sqrt{\sigma}} = 0.631 \pm 0.002 \quad . \tag{6}$$

The reduction of the cut-off dependence may best be discussed by comparing calculations of $T_c/\sqrt{\sigma}$ at a fixed value of the cut-off, for instance for $aT_c = 1/4$, with the above continuum extrapolation. Such a comparison is performed in Table 1 for some tree level and tadpole improved actions as well as the RG-improved action. While in the case of the Wilson action $T_c/\sqrt{\sigma}$ deviates by roughly 10% from the continuum extrapolation at this value of the cut-off, the tree level and tadpole improved actions only show little deviations. We also note that the result for the tadpole improved action is consistent within errors with the tree level result. The RG-improved action does lead to a somewhat larger value for $T_c/\sqrt{\sigma}$ than all the other calculations presented in Table 1. However, when comparing calculations at different values of the cut-off (Fig. 2) we note that the RG-improved action as well as the other improved actions shows no significant cut-off dependence, unlike the Wilson action

[c] We use the notation FP I and FP III for the two sets of fixed point actions defined in [15].

[d] This differs slightly from the value published in [19]. The difference results from a re-analysis of the string tension [20] at $\beta_c(N_\tau)$ for $N_\tau = 8$ and 12 which now is based on an interpolation of results obtained at nearby values of the gauge coupling rather than a single calculation performed at $\beta_c(N_\tau)$ as it has been done in [19].

Table 1
Critical temperature in units of $\sqrt{\sigma}$ on lattices with temporal extent $N_\tau = 4$, *i.e.* at a value of the cut-off given by $aT_c = 0.25$. Infinite volume extrapolations for the critical couplings have been performed in all cases. Further details on the data can be found in the references given in Figure 2.

action	β_c	$T_c/\sqrt{\sigma}$
standard Wilson	5.69254 (24)	0.5983 (30)
(2,2) (tree level improved)	4.3999 (3)	0.625 (4)
(1,2) (tree level improved)	4.0730 (3)	0.633 (3)
(1,2) (tadpole improved)	4.3525 (5)	0.634 (3)
(1,2) (RG improved)	2.2879 (11)	0.653 (6)(1)

where the $\mathcal{O}(a^2)$ cut-off dependence is clearly visible. It thus seems that the discrepancy of about 4% between the different improved action calculations is mainly due to differences in the determination of the string tension from the heavy quark potential and not due to different $\mathcal{O}(a^2)$ corrections, *i.e.* the accuracy is at present limited through ambiguities in extracting the long-distance physics from the heavy quark potential. This ambiguity is reflected in the spread of current determinations of the critical temperature,

$$T_c/\sqrt{\sigma} = 0.631\ (2)\ [19,20] - 0.656\ (4)\ [14].$$

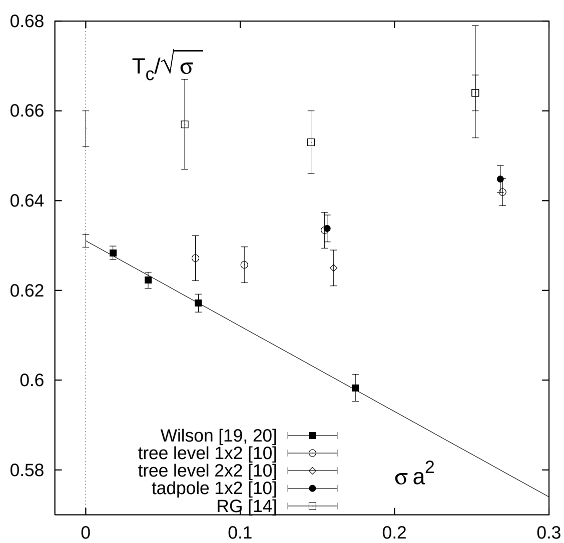

Figure 2. The critical temperature in units of the square root of the string tension for various actions versus the square of the cut-off. For the tree level improved $(1,2)$-action results for the critical coupling on finite spatial lattices with temporal extent $N_\tau = 5$ and 6 have been taken from Ref [21] and extrapolated to infinite volume [10].

4.2. Bulk thermodynamics

Bulk thermodynamic quantities like the energy density (ϵ) or pressure (p) calculated with the Wilson action on lattices of size $N_\sigma^3 N_\tau$ with $N_\tau = 4$, 6 and 8 [19] show a strong cut-off dependence in the plasma phase. A first analysis with the tree level improved (2,2)-action [9] has shown that this cut-off dependence gets drastically reduced already on an $N_\tau = 4$ lattice. This is true also for the tree level improved (1,2)-action, the tadpole improved (1,2)-action [11] and a fixed point action [15] which all yield results consistent with the continuum extrapolation performed for the standard Wilson action. As an example we discuss here the pressure.

The pressure can be obtained from an integration of the difference of action densities at zero (S_0) and finite (S_T) temperature [22],

$$\frac{p}{T^4}\Big|_{\beta_0}^{\beta} = N_\tau^4 \int_{\beta_0}^{\beta} \mathrm{d}\beta' (S_0 - S_T) \ , \qquad (7)$$

where the zero temperature calculations are performed on a large lattice of size N_σ^4 and the finite temperature calculations are performed on lattices of size $N_\sigma^3 N_\tau$. In order to compare calculations performed with different actions one has to determine a physical temperature scale. This can, for instance, be achieved through a calculation of the string tension at several values of the gauge coupling (β) and at the critical coupling for the phase transition on a lattice of temporal size N_τ. This yields $T/T_c \equiv \sqrt{\sigma}(\beta_c)/\sqrt{\sigma}(\beta)$.

A systematic analysis of the cut-off dependence of bulk thermodynamic observables has been performed previously for the Wilson action [19]. Calculations on lattices with temporal extent $N_\tau =$ 4, 6 and 8 could be used there to perform an extrapolation to the continuum limit. In Fig. 3 we show

the results of a calculation of the pressure using various improved actions on $N_\tau = 4$ lattices. The strong cut-off dependence for the Wilson action is clearly seen in the upper plot. While all calculations with improved actions are close to the continuum extrapolation obtained from the Wilson action (solid line), the Wilson action result obtained directly on an $N_\tau = 4$ lattice shows strong deviations (upper curve). In fact, it seems that the scattering of the various improved action calculations around the continuum extrapolation mainly reflects the uncertainties in fixing the temperature scale through calculations of the string tension rather than systematic differences in the cut-off dependence. This is also supported by the lower part of Fig. 3 where we show a comparison of improved action calculations at different values of the cut-off. Calculations on $N_\tau = 3$ and 4 lattices do not show any significant cut-off dependence for the tree level as well as for the tadpole improved actions. On the other hand, the spread of results obtained with different actions is larger than the cut-off dependence visible for a given improved action. The results also agree well with a calculation of the pressure performed with a fixed point action on lattices of size $12^3 \times 3$ (triangles) [15]. It thus seems that systematic errors in the calculation of the equation of state which result from the lattice discretization are well under control for the SU(3) gauge theory.

In Fig. 4 we show the energy density, pressure and entropy density obtained from an extrapolation of results obtained with the standard Wilson action. We note the rapid rise in all observables at T_c followed by a rather slow approach to the asymptotic ideal gas behaviour. The latter is in accordance with the expectation that the high temperature behaviour of QCD is controlled by a universal function, which only depends on a *running coupling* that varies logarithmically with temperature. However, although ϵ, s and p rapidly come close to the ideal gas limit, the (15-20)% deviations observed at temperatures as large as $5T_c$ are still too large to be described by perturbation theory. The perturbative expansion of the thermodynamic potential is converging quite badly and would require a running coupling which is significantly smaller than unity [23,24]. As a consequence, the perturbative expansion seems to converge only for temperatures much larger than T_c. In particular, the calculation of screening lengths on the lattice do, however, suggest that the running coupling is larger than unity even at $5T_c$ [25].

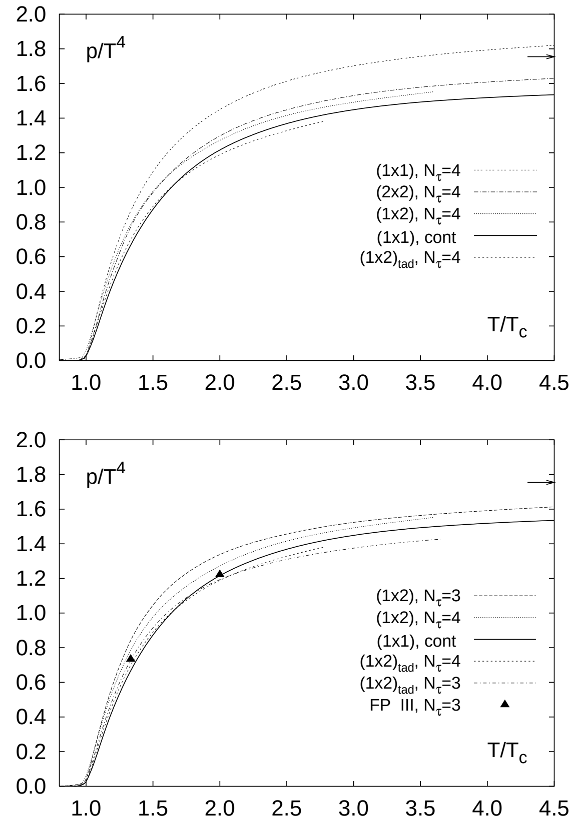

Figure 3. Pressure of the SU(3) gauge theory calculated with the Wilson action and different improved actions on $N_\tau = 4$ lattices (upper figure). The lower figure shows a comparison of calculations with tree level and tadpole improved actions on $N_\tau = 3$ and 4 lattices. Also shown there are results from a calculation with a fixed point action (triangles). The arrows indicate the ideal gas result in the continuum limit.

4.3. Surface tension and latent heat

The success of improved actions for the calculation of bulk thermodynamics even at temperatures close to T_c naturally leads to the question whether these actions also do lead to an improvement at T_c itself where a strong cut-off dependence has been observed previously in the calculation of the latent heat ($\Delta\epsilon$) and the surface tension (σ_I) in studies with the Wilson action on lattices with temporal extent up to $N_\tau = 6$ [26,27]. The region around T_c is, of course, a highly non-perturbative regime. However, observables like $\Delta\epsilon$ and σ_I, which characterize the discontinuities at the first order deconfinement phase transition in a $SU(3)$ gauge theory,

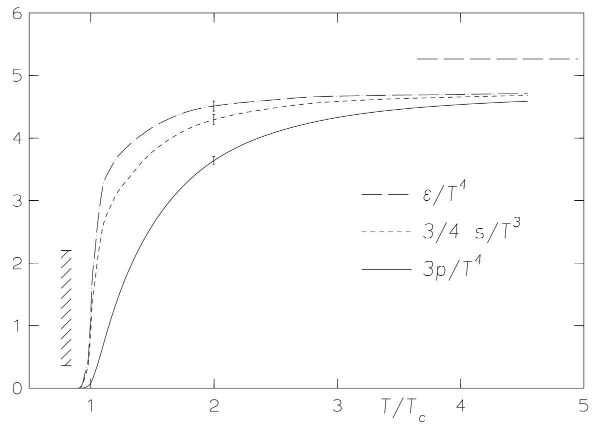

Figure 4. Extrapolation to the continuum limit for the energy density, entropy density and pressure versus T/T_c. The dashed horizontal line shows the ideal gas limit. The hatched vertical band indicates the size of the discontinuity in ϵ/T^4 (latent heat) at T_c. Typical error bars are shown for all curves.

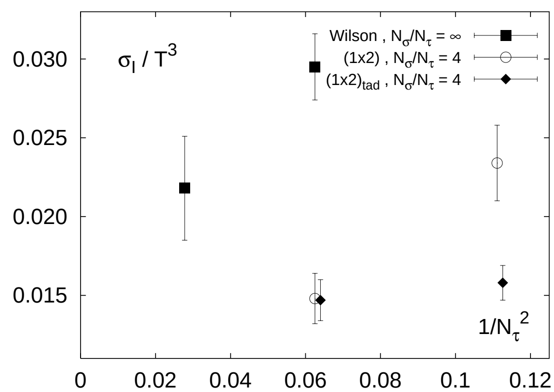

Figure 5. Comparison between surface tension calculations performed with the Wilson action as well as tree level and tadpole improved (1,2)-actions.

do depend on properties of the low as well as the high temperature phase. As the latter is largely controlled by high momentum modes it may be expected that some improvement does result even from tree level improved actions.

We have extracted σ_I [11] from the probability distribution of the absolute value of the Polyakov loop, $P(|L|)$, following the analysis presented in Ref. [27]. The probability distribution at the minimum is proportional to

$$P(|L|) \sim \exp\bigl(-\bigl[f_1 V_1 + f_2 V_2 + 2\sigma_I A\bigr]/T\bigr) \qquad (8)$$

where f_i denotes the free energy in the phase i, V_i is the volume occupied by that phase and A denotes the interface area of the finite system. The distribution functions for Polyakov loop at T_c show a double peak structure. From the depth of the minimum between these peaks one can extract the surface tension.

In Fig. 5 we show results for the surface tension calculated with the Wilson action as well as tree level and tadpole improved (1,2)-actions at different values of the cut-off. It is quite remarkable that the results for the tadpole improved action show practically no cut-off dependence when comparing calculations on $N_\tau = 3$ and 4 lattices. Furthermore it is evident that both improved actions lead to significantly smaller values for the surface tension at finite values of the cut-off. In fact, the results for $N_\tau = 4$ are in agreement with a quadratic extrapolation of the Wilson data to the continuum limit.

In Table 2 we give results for σ_I on the largest lattices considered. Clearly the surface tension extracted from simulations with improved actions on lattices with temporal extent $N_\tau = 4$ are substantially smaller than corresponding results for the Wilson action. In fact, they are compatible with the $N_\tau = 6$ results for the Wilson action. While the cut-off dependence of the surface tension calculated on finite lattices seems to be well under control now through the use of improved actions, it remains to get control over the behaviour in the thermodynamic limit ($N_\sigma \to \infty$). The difference between the surface tension calculated on the largest spatial lattice and the extrapolated value is about 20%. However, it is not quite clear whether the ansatz used to perform the extrapolation to the infinite volume is already applicable for the currently used spatial lattice sizes. It includes, for instance, the contribution of translational zero modes, which are not yet observed in the present calculations [27]. Given these caveats we find from an extrapolation of the results obtained with the tadpole improved actions [11]

$$\frac{\sigma_I}{T_c^3} = 0.0155 \pm 0.0016 \qquad (9)$$

The latent heat is calculated from the disconti-

Table 2
Surface tension and latent heat for improved actions and the Wilson action. Results for the Wilson action are based on data from [27] using the non-perturbative β-function calculated in [19]. Improved action results are taken from [11].

action	V_σ	N_τ	σ_I/T_c^3	$\Delta\epsilon/T_c^4$
standard Wilson	$24^2\times 36$	4	0.0300 (16)	2.27 (5)
	$36^2\times 48$	6	0.0164 (26)	1.53 (4)
(1,2)-action (tree level)	32^3	4	0.0116 (23)	1.57 (12)
(1,2)-action (tadpole)	32^3	4	0.0125 (17)	1.40 (9)

nuity in $(\epsilon-3p)$. This in turn is obtained from the discontinuity in the various Wilson loops entering the definition of the improved actions. In the case of actions, which are defined through a set of couplings, which do not depend on the gauge coupling g^2, the latent heat is, in fact, simply proportional to the discontinuity in the action expectation value at T_c. In general one has, however, to take into account contributions resulting from derivatives of the couplings $c_i(g^2)$ with respect to g^2. The latent heat is then given by

$$\frac{\Delta\epsilon}{T_c^4} \quad = \quad \frac{1}{6}\left(\frac{N_\tau}{N_\sigma}\right)^3\left(a\frac{\mathrm{d}\beta}{\mathrm{d}a}\right)\left(\langle\tilde{S}\rangle_+ - \langle\tilde{S}\rangle_-\right) \qquad (10)$$

with $\tilde{S}\equiv S-\mathrm{d}S/\mathrm{d}\beta$.

The discontinuity in the expectation value of $\tilde{S}$ at β_c is obtained by calculating these separately in the two coexisting phases at β_c. In order to extract the latent heat one still needs the β-function entering the definition of $\Delta\epsilon/T_c^4$ in Eq. 10. The necessary relation $a(\beta)$ has been obtained from a calculation of $\sqrt{\sigma}a$ (improved actions) or a determination of $T_c a$ (Wilson action) [11]. Results for $\Delta\epsilon/T_c^4$ on $N_\tau=4$ and 6 lattices are summarized in Table 2.

A comparison with simulations on even coarser lattices ($N_\tau=3$) [11] shows that the cut-off dependence in the latent heat follows a similar pattern as that shown for the surface tension in Fig. 5. In particular, the results obtained with the tadpole improved action on $N_\tau=3$ and 4 lattices agree with each other within errors. We thus may use these results to estimate the latent heat in the continuum limit. Using Eq. 6 and $\sqrt{\sigma}=420$ Mev we find

$$\Delta\epsilon\simeq 1.5T_c^4 = 0.23\sigma^2 = 0.9\ \mathrm{GeV/fm}^3 \qquad (11)$$

5. Four-flavour QCD with an improved staggered action

In the fermionic sector of QCD the influence of a finite cut-off on bulk thermodynamic observables is known to be even larger than in the pure gauge sector. For instance, in the staggered formulation the energy density of an ideal fermi gas differs by more than 70% from the continuum value on a lattice with temporal extent $N_\tau=4$ and approaches the continuum value only very slowly with increasing N_τ. This cut-off dependence can drastically be reduced with an $O(a^2)$ improved staggered action. In a first attempt to analyze the importance of improvement in the fermion sector we have performed calculations with an improved action, $S^I[U]=S^{(1,2)}+\bar{\psi}M\psi$, where a higher order difference scheme (one-link and three-link terms), is used to improve the fermionic part [28]. The improved fermion matrix reads

$$M[U]_{xy} = m\delta_{x,y} + \left(\frac{9}{16}A\,[U]_{x\ y} - \frac{1}{48}B\,[U]_{x\ y}\right)$$

with

$$A[U]_{xy} = \sum_\mu \eta_{x,\mu}\left(U_{x,\mu}\delta_{x,y-\hat{\mu}} - U^\dagger_{x-\hat{\mu},\mu}\delta_{x,y+\hat{\mu}}\right)$$

$$B[U]_{xy} = \sum_\mu \eta_{x,\mu}(U_{x,\mu}U_{x+\hat{\mu},\mu}U_{x+2\hat{\mu},\mu}\delta_{x,y-3\hat{\mu}} - U^\dagger_{x-\hat{\mu},\mu}U^\dagger_{x-2\hat{\mu},\mu}U^\dagger_{x-3\hat{\mu},\mu}\delta_{x,y+3\hat{\mu}})$$

where $\eta_{x,\mu}=(-1)^{x_0+..+x_{\mu-1}}$ and $\eta_{x,0}\equiv 1$ denote the staggered phase factors. In the gluonic sector we use the tree level improved (1,2)-action. With this action the overall cut-off distortion of the ideal gas limit on a $16^3\times 4$ lattices reduces to about 20%. We have performed simulations for two quark masses, $ma=0.05$ and 0.1 [17]. Like in the pure gauge sector the improvement is visible already close to T_c.

The general structure of the four-flavour equation of state does not differ much from that of the pure gauge theory. In fact, the temperature dependence of the pressure is very similar to that of the pure gauge theory, if we re-scale the latter by an appropriate ratio of the number of degrees of freedom so that the high temperature limit coincides for both cases. This is shown in Fig. 6.

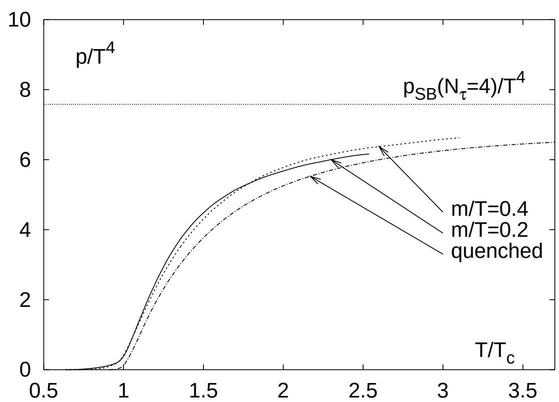

Figure 6. Comparison between the pressure of four-flavour QCD on a $16^3 \times 4$ lattice for two values of the quark mass and the pressure of the pure $SU(3)$ gauge theory. The latter has been re-scaled by the appropriate number of degrees of freedom of four-flavour QCD (29/8) and a factor 1.19 which takes care of the remaining cut-off distortion of the ideal gas limit resulting from the use of the Naik action.

While the pressure is obtained in complete analogy to the pure gauge theory through an integration of action differences (Eq. 7), the calculation of the energy density now also requires the determination of differences of the chiral condensates at zero and non-zero temperature as well as the cut-off dependence of the two bare couplings, β and m_q,

$$\frac{\epsilon - 3p}{T^4} = -N_\tau^4 \Bigg[\frac{\mathrm{d}\beta}{\mathrm{d}\ln a} \Big(S_0 - S_T \Big) + \frac{\mathrm{d}ma}{\mathrm{d}\ln a} \Big(\langle \bar{\chi}\chi \rangle_0 - \langle \bar{\chi}\chi \rangle_T \Big) \Bigg] \qquad (12)$$

Only in the chiral limit the derivative $\mathrm{d}\ln ma/\mathrm{d}a$ vanishes and $(\epsilon - 3p)$ is again proportional only to the β-function, $\mathrm{d}\beta/\mathrm{d}\ln a$, as it is the case in the pure gauge sector.

Results for the energy density are shown in Fig. 7. The energy density does stay close to the ideal gas limit immediately above T_c. We do observe an overshooting of the ideal gas limit close to T_c for the non-zero quark masses considered by us. This is a feature not seen before in the pure gauge sector. Whether this will persist for finite values of the quark mass or is an artifact of our present statistical accuracy has to be clarified in further more detailed investigations. The overshooting seems, however, to disappear in an "extrapolation" to the chiral limit, which we constructed by ignoring the term proportional to $\langle \bar{\psi}\psi \rangle$ in the definition of the energy density [17]. The contribution of this term vanishes in the chiral limit as the derivative $\mathrm{d}\ln ma/\mathrm{d}a$ is proportional to the quark mass.

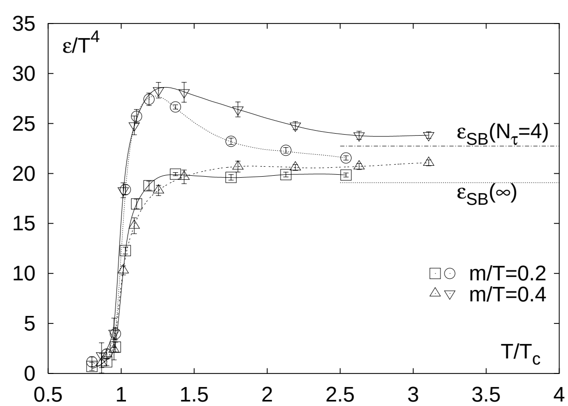

Figure 7. Energy density of four-flavour QCD on a $16^3 \times 4$ lattice. The lower set of curves shows an "extrapolation" to the chiral limit which has been obtained by ignoring the second term in Eq. 10 (see text).

We note that the energy density in the critical region is larger than in the pure gauge case when expressed in units of T_c, *i.e.* $\epsilon \sim 10 T_c^4$. However, as the critical temperature is substantially smaller than in the pure gauge theory, the critical energy density again turns out to be similar, *i.e.* about 1 GeV/fm^3.

6. Improved Fermion Actions

The calculations with an improved fermion action show that a strong reduction of the cut-off dependence is possible in the high temperature phase.

Table 3
Coefficients for rotationally invariant staggered fermion actions

(j,k,l,m)	$c_{j,k,l,m}$	
	p4-action	p6-action
(1,0,0,0)	0.375	0.32
(1,2,0,0)	0.0208333	0.02
(1,2,2,0)		0.0010938
(3,0,0,0)		0.0047917
(1,2,2,2)		0.00125
(3,2,0,0)		0.00125

Still a further improvement is necessary in order to reduce the cut-off dependence to only a few percent as it is the case in the pure gauge sector.

Although the Naik action significantly reduces the cut-off dependence of thermodynamic quantities calculated with the staggered fermion action the deviations from the continuum result are still about 20% on lattices with temporal extent $N_\tau = 4$. Moreover it seems that the flavour symmetry, which is broken at $\mathcal{O}(a^2)$ in the staggered fermion action, is not much improved in the Naik formulation [30]. It thus is desirable to look for more suitable fermion actions. In general, we may discretize the free fermion action not only by introducing nearest neighbour couplings between fermion fields but we may also allow for couplings between sides further apart,

$$\begin{aligned} S_F \;=\; & \sum_{x,\mu} \eta_{x,\mu} \bar{\psi}(x) \sum_{j>0,k,l,m} c_{j,k,l,m} \cdot \\ & \cdot \big[\psi(x + ja_\mu + ka_\nu + la_\rho + ma_\sigma) \\ & - \psi(x - ja_\mu - ka_\nu - la_\rho - ma_\sigma)\big] \ . \end{aligned} \tag{13}$$

A specific choice of couplings $c_{j,k,l,m}$ has been given in [29] for a staggered fixed point action. This does, however, include too many non-zero couplings for being useful in a numerical simulation.

From the analysis of the improved gluon actions involving six-link operators we learned that non-planar loops are, in fact, more efficient in reducing the cut-off dependence, even when the $\mathcal{O}(a^2)$ corrections are not eliminated exactly. One thus may consider to replace the straight three-link term in the Naik action by a bended three link term. The coefficient of this term can be fixed by demanding that the fermion propagator is rotationally invariant up to $\mathcal{O}(p^4)$. Similarly one can construct actions which include longer paths and are rotationally invariant up to $\mathcal{O}(p^6)$. We call these actions p4 and p6-action, respectively. For instance, in order to obtain a fermion propagator, which is rotationally invariant up to $\mathcal{O}(p^4)$ within the class of staggered fermion actions that contain non-zero couplings only for one and three-link paths, the coefficients are constrained by

$$\begin{aligned} c_{1,0,0,0} + \ 3c_{3,0,0,0} + 6c_{1,2,0,0} &= \frac{1}{2} \\ c_{1,0,0,0} + 27c_{3,0,0,0} + 6c_{1,2,0,0} &= 24c_{1,2,0,0} \ . \end{aligned} \tag{14}$$

Similar constraints can be obtained for actions that are rotationally invariant up to $\mathcal{O}(p^6)$. A possible set of non-zero couplings for these actions is given in Table 3 and a calculation of the fermion energy density is displayed in Fig. 8. We note that with the simple choice of the p4-action, which includes in addition to the standard one-link term only a bended three-link term, the cut-off dependence is reduced to a few percent even on $N_\tau = 4$ lattices. As the introduction of gauge fields in this action will require smeared paths like in the fat link action proposed in [31] there is the possibility that such actions also improve the flavour symmetry of the staggered fermion formulation. This, however, has to be examined still in more detail.

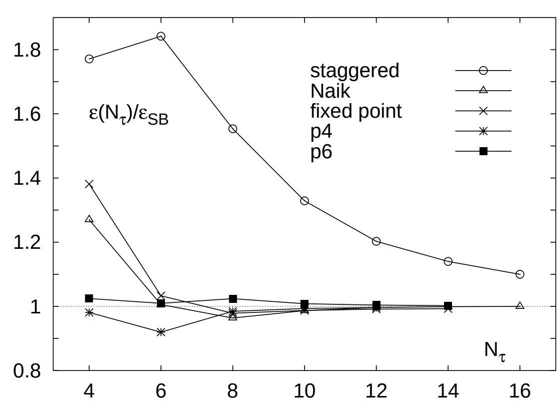

Figure 8. Cut-off dependence of the fermion energy density calculated from several improved actions.

7. Conclusions

Thermodynamic observables of the $SU(3)$ gauge theory and QCD studied with improved gauge and

fermion actions show a drastic reduction of the cut-off dependence in the high temperature limit as well as at T_c. The major improvement effect is already obtained with tree level improved actions. The calculation of the equation of state of the $SU(3)$ gauge theory seems to be well under control and major sources for lattice artifacts have been eliminated. At least for finite temperature calculations on lattices with temporal extent $N_\tau = 4$ or larger systematic effects due to additional tadpole improvement are within the current statistical accuracy. Tadpole improvement leads to further reduction of systematic errrors on $N_\tau = 3$ lattices. This opens the possibility to repeat quantitative studies of various other thermodynamic quantities on rather coarse lattices which so far could only be investigated on a qualitative level without reliable extrapolations to the continuum limit.

The calculation of thermodynamic quantities with improved fermion actions has just started. In this case further work is still needed in order to be able to select an appropriate action which even in the ideal gas limit has as little cut-off dependence as the tree level improved gluon actions.

Acknowledgments: The work reported here has been performed in collaboration with my colleagues at the University of Bielefeld. I would like to thank them for many discussions and their help in preparing this talk. In particular I would like to mention B. Beinlich, J. Engels, R. Joswig, E. Laermann, C. Legeland, M. Lütgemeier, A. Peikert and B. Petersson.

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